

ASPECTS OF PLANE WAVE (MATRIX) STRING DYNAMICS

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We analyse two issues that arise in the context of (matrix) string theories in plane wave backgrounds, namely (1) the use of Brinkmann- versus Rosen-variables in the quantum theory for general plane waves (which we settle conclusively in favour of Brinkmann variables), and (2) the regularisation of the quantum dynamics for a certain class of singular plane waves (discussing the benefits and limitations of regularisations of the plane-wave metric itself).

1 INTRODUCTION

As exact and potentially exactly solvable string backgrounds, plane waves have been extensively studied in string theory (see e.g. [1] for a review of the early literature). As such they also provide an ideal setting for studying the issue and fate of time-dependent backgrounds and space-time singularities in string theory [2, 3]. In the usual Brinkmann coordinates, a plane wave metric has the form

$$ds^2 = 2dz^+dz^- + A_{ab}(z^+)z^az^b(dz^+)^2 + \delta_{ab}dz^adz^b \ ,$$

and this metric is singular iff the profile $A_{ab}(z^+)$ (= minus the matrix of frequency-squares for transverse string modes in the lightcone gauge $z^+ \sim t$) diverges for some (finite) value of z^+ .¹ Among all singular plane waves, from the point of view of studying string theory in singular backgrounds, those with a profile of the form $A_{ab}(z^+) \sim (z^+)^{-2}$ are singled out by at least two facts:

1. Precisely these backgrounds were shown in [5, 6, 7] to arise universally as the Penrose limits [8, 9, 10] of a large general class of singularities (of power-law type [11]) which encompasses all the standard string theory black hole and black brane singularities as well as those of all (non-oscillatory/chaotic) standard cosmological models. Thus these special singular plane waves are not just toy-models of space-time singularities but honest 1st-order approximations (in the spirit of the Penrose-Fermi expansion [12] around the Penrose limit metric) to realistic types of singularities.
2. These singular plane waves possess precisely those symmetries (isometries and homotheties), in particular the boost/scale symmetry $z^\pm \rightarrow \lambda^\pm z^\pm$, that are required [13] to implement the usual (flat space) Seiberg-Sen(-CSV) [14, 15, 16, 17] DLCQ derivation of matrix string theory [18] for these curved backgrounds, so that in principle one has available a non-perturbative definition of string theory in these backgrounds (the plane wave matrix big bang models of [13]).

The above-mentioned boost/scale symmetry renders the metric homogeneous away from the locus of the singularity at $z^+ = 0$ (see e.g. [10, 19, 20]). Thus these metrics can be considered as singular counterparts of the much studied (symmetric) plane waves with constant profile A_{ab} (which are homogeneous because of the invariance under lightcone time translations), and this scale-invariance is reflected in (and impacts upon) the quantisation of strings and particles in such a background [19, 20, 21, 22]. In the lightcone gauge $z^+ = t$, the string mode equations are simply harmonic oscillator equations with time-dependent frequencies $\omega_n(t)^2 = \omega(t)^2 + n^2$, with $\omega(t)^2 \sim t^{-2}$ arising from the plane wave profile. In particular, perturbative string theory in these singular homogeneous plane waves (with weak string coupling at the singularity) was analysed in detail in [19], with the conclusion that, although there is divergent particle production on the world-sheet as one approaches the singularity at $t = 0$ (cf. also [2] and [23]), there is a possibility

¹What is meant by a singularity of the metric in the present context (where all scalar curvature invariants of the metric are zero) is divergent tidal forces, manifested by a divergence of components of the Riemann tensor in a parallel propagated frame (p.p.-singularities in the terminology of [4]). Thus plane waves with a singular profile have null p.p.-singularities.

that an analytic continuation of the string wave-functions could lead to non-singular evolution through the singularity.

Curiously exactly the same (decoupled, Abelian) string mode equations arise from the analysis of strong string coupling singularities in the context of the (non-Abelian) matrix string models [13] for singular plane waves, in which the strong string coupling implies weak (time-dependent) Yang-Mills gauge coupling. This comes about because, as shown in [24], near the singularity the usual quartic interaction term for the non-Abelian matrix string coordinates becomes irrelevant.² Thus the near-singularity behaviour of the matrix string coordinates is governed only by the decoupled (non-self-interacting) kinetic and harmonic oscillator (mass) terms $\sim \omega_n(t)^2$, as above, and one formally encounters the same problems and issues as in the perturbative string theory model of [19], now however at strong string coupling.

The (thus necessary) issue of regularisation of the particle and string dynamics in these singular backgrounds was taken up again in [25, 26].³ In particular, in [26] a specific proposal was made to regularise the dynamics, namely by an explicit single-scale regularisation of the oscillator frequencies (equivalently, of the plane wave metric) of the form

$$\omega^2(t) \rightarrow \Omega_\epsilon^2(t) = \epsilon^{-2} \Omega^2(t/\epsilon)$$

in which the scale invariance of the original background is as much as possible respected. In [26, 28] some minimal conditions were proposed for such a regularisation to lead to a well-defined quantum evolution in the limit $\epsilon \rightarrow 0$ that the regulator is removed. One of the purposes of this article is to analyse and scrutinise this proposal in some more detail.

Before embarking on this, however, there is another issue that needs to be addressed and settled once and for all which is a minor annoyance but which occasionally rears its ugly head. This is the question whether one should analyse or quantise a plane wave system in the Brinkmann coordinates used above, or whether one can (either alternatively or even profitably) study the system in variables based on Rosen coordinates, in which a plane wave metric takes the form

$$ds^2 = 2dudv + g_{ij}(u)dx^i dx^j \ .$$

In particular, in the present context, metrics of this form appear to provide attractive models of *null cosmologies* [19]. However, as is well known (and as we will recall at the beginning of section 2), these Rosen coordinate systems are fraught with ambiguities, almost invariably exhibiting spurious coordinate singularities, while the Brinkmann coordinate system encodes directly geometrically invariant and physical information about the space-time (coordinates z^a are measures of proper distance, and the profile A_{ab} assembles the parallel orthonormal frame components of the Riemann tensor).

Since it has, in spite of this, occasionally been suggested (see e.g. [29, 30]) that the Rosen system may provide a better, because seemingly less singular, description of the quantum dynamics, in section 2 we first quickly go through the (essentially undergraduate) exercise of constructing

²This is a stronger statement than just the fact that the Yang-Mills coupling constant goes to zero, its validity relying [24] on the precise time-dependence of the dilaton arising from the requirement that the metric-dilaton configuration be a solution to the string background equations of motion [13].

³See also the review [27] for a discussion of the state of affairs.

and solving the quantum theory of Brinkmann and Rosen particles and the (unitary) relation between them. We then revisit the Brinkmann vs Rosen issue in this quantum context, stress that this is not a question of Rosen vs Brinkmann quantum states but rather a matter of which operators one attributes physical significance to, and show that the plane wave space-time origin of these models uniquely leads to the Brinkmann picture. This conclusion is then significantly strengthened and sharpened in section 2.4, where we show that some rather remarkable and striking recent uniqueness results [31, 32, 33, 34, 35] regarding the quantisation of scalar fields with time-dependent mass terms imply that for strings in arbitrary plane wave backgrounds (1) in the Brinkmann parametrisation there is a unique Fock quantisation respecting the translation invariance along the worldsheet S^1 (namely the standard Fock space of a free massless scalar field) such that the dynamics is unitarily implemented; and (2) that there is no S^1 -invariant Fock quantisation of the Rosen system with unitary implementable time evolution (in the singular case, with a mass term $\sim t^{-2}$, these results hold for any $t \neq 0$).

The upshot of this discussion is that the quantisation of (matrix) strings in plane wave backgrounds needs to be performed in Brinkmann variables (something that we had already advocated in [13] on the related grounds that it is only in these variables that the scalars have a canonical kinetic term), and that in spite of the time-dependence of the mass-term / Hamiltonian (which usually leads to quantisation ambiguities and phenomena like particle production etc.) there is a unique Fock quantisation with unitary time-evolution.

Having disposed of the Rosen representation, we then (re-)turn to the issue of regularisation of the dynamics in singular scale-invariant plane waves and the proposal of [26]. We first specialise the results of section 2 regarding expectation values to the singular plane waves to illustrate how the plane wave singularity manifests itself in the quantum theory. We then exhibit a family of single-scale regularisations that satisfy the necessary conditions of [26, 28], essentially by a method that amounts to directly regularising the Rosen metric rather than the Brinkmann wave profile / frequency. We analyse the extent to which such regularisations can be considered to give a non-singular evolution through $t = 0$ in the limit that the regularisation parameter $\epsilon \rightarrow 0$, with the conclusion that no regularisation of the single-scale form can provide this. More complicated regularisations that achieve this do exist, however, and we comment on this as well as on issues related to the regularisation of the dilaton. We end with a discussion of the results, in particular their implications for matrix string models of plane wave null singularities, and their interpretation in terms of the Penrose limit origin of these space-times.

2 QUANTUM BRINKMANN VS ROSEN PLANE WAVE DYNAMICS

In this section we will analyse the relation between the quantum dynamics of particles and strings in the Brinkmann and Rosen parametrisations. To set the stage and motivate the discussion of this section, we begin by recalling the well-known ambiguities and spurious singularities afflicting the Rosen description of the classical particle dynamics.

2.1 CLASSICAL DYNAMICS AND ROSEN SINGULARITIES

In Brinkmann (resp. Rosen) coordinates, plane wave metrics have the form

$$\begin{aligned} ds^2 &= 2dz^+dz^- + A_{ab}(z^+)z^az^b(dz^+)^2 + \delta_{ab}dz^adz^b \\ &= 2dudv + g_{ij}(u)dx^idx^j \end{aligned} \quad (2.1)$$

($u = z^+$, the remaining coordinate transformation between the unique Brinkman form of the metric and the non-unique Rosen form is given in (A.3)). For present purposes nothing is lost by considering the isotropic (or transverse 1-dimensional) case $g_{ij}(u) = e(t)^2\delta_{ij}$ and correspondingly $A_{ab}(u) = A(u)\delta_{ab}$.⁴ The Brinkmann and Rosen Lagrangians for a relativistic particle in the lightcone gauge $z^+ = u = t$ are

$$L_{bc}(z, \dot{z}; t) = \frac{1}{2}(\dot{z}^2 - \omega^2(t)z^2) \quad L_{rc}(\dot{x}; t) = \frac{1}{2}e(t)^2\dot{x}^2, \quad (2.2)$$

the general relation (A.4) between the Rosen and Brinkmann parameters g_{ij} and A_{ab} reducing to the harmonic oscillator equation

$$A(t) = \frac{\ddot{e}(t)}{e(t)} \equiv -\omega^2(t) \quad \text{or} \quad \ddot{e}(t) + \omega^2(t)e(t) = 0. \quad (2.3)$$

Up to a total derivative term, L_{bc} and L_{rc} are related by the coordinate transformation $z = e(t)x$, the former being exactly the Lagrangian of a time-dependent harmonic oscillator, the latter that of a “free” particle with time-dependent mass. The non-uniqueness of Rosen coordinates, $x = e(t)^{-1}z$, and of the Rosen Lagrangian, is reflected in the arbitrary choice of solution $e(t)$ of the oscillator equations of motion (2.3). The corresponding Hamiltonians,

$$H_{bc} = \frac{1}{2}(p_z^2 + \omega(t)^2z^2) \quad H_{rc} = \frac{1}{2}e(t)^{-2}p_x^2, \quad (2.4)$$

are then related by a linear time-dependent canonical transformation, the only (barely) non-trivial feature being that as a consequence of this time-dependence the Hamiltonians differ not only by the canonical transformation of the canonical variables but also by the time-derivative of the corresponding generating function (A.8).

Brinkmann coordinates provide a global coordinate chart for plane wave space-times while Rosen coordinates typically (and almost invariably, e.g. for backgrounds satisfying the weak energy condition, see [36] and Appendix A.1) exhibit spurious coordinate singularities. As a prototypical example, consider the case of constant A_{ab} i.e. constant frequency ω . In Rosen coordinates one attempts to use the classical trajectories of the Brinkmann system as Cartesian coordinate lines (as reflected in the fact that $x^k = \text{const.}$ is always a solution of the Rosen equations of motion), which in the present case obviously doesn’t bode well for these coordinates as the oscillator reaches its turning point. Indeed, with the choice $e(t) = \sin \omega t$, say, the Rosen metric and Hamiltonian have the form

$$ds^2 = 2dudv + \sin^2 \omega t dx^2 \quad H_{rc} = \frac{1}{2}(\sin^2 \omega t)^{-1}p_x^2 \quad (2.5)$$

⁴In particular, such metrics can always be elevated to full-fledged string theory background by a suitable choice of null dilaton.

which is evidently not able to describe the perfectly regular evolution of the system (or dynamics of strings in e.g. the maximally supersymmetric IIB background [37, 38, 39, 40, 41]) beyond $t = \pi/\omega$.

The situation may at first appear to be less clear-cut in the case of genuinely singular plane waves in which both Rosen and Brinkmann metrics exhibit a singularity at, say, $u = 0$. As mentioned in the introduction, Rosen metrics are occasionally regarded as null analogues of cosmological metrics. For instance, isotropic metrics with $g_{ij}(u) = g(u)\delta_{ij}$ look like null counterparts of standard spatially flat ($k = 0$) FRW metrics, and metrics with a power-law behaviour $g_{ij}(u) = u^{2a_i}\delta_{ij}$ are null analogues of Kasner metrics. The latter, incidentally, are precisely the Rosen forms of the scale-invariant plane waves for a certain range of frequencies,

$$g_{ij}(u) = u^{2a_i}\delta_{ij} \quad \Rightarrow \quad A_{ij}(z^+) = a_i(a_i - 1)\delta_{ij} (z^+)^{-2} . \quad (2.6)$$

However, interpreting Rosen plane waves in this way requires some care. For example, consider the simple (both isotropic and power-law) metric with $g(u) = u^4$. This form of the metric suggests that it describes a null version of a Big Bang at $u = 0$. However, the very same metric can be written in a different coordinate system as a Rosen plane wave with $\tilde{g}(u) = u^{-2}$, with the same u , suggesting that what happens at $u = 0$ is a Big Rip, not a Big Bang. Clearly then, the connection between the form of the metric and the physics and geometry it actually describes is at best somewhat hidden in Rosen coordinates. It is however completely manifest in Brinkmann coordinates (in which e.g. both the above models give rise to a scale-invariant plane wave with $\omega^2(t) = -2t^{-2}$), since Brinkmann coordinates are Fermi coordinates along the family of null geodesics with $z^a = 0$ and $z^- = \text{const.}$ [12]. In particular, the transverse coordinates z^a are Riemann normal coordinates and thus a direct measure of physical proper distance in the space-time, and the wave profile $A_{ab}(z^+) = -R_{+a+b}$ gives the (only non-vanishing) components of the Riemann tensor in a parallel propagated frame along this null geodesics, and thus directly encodes the only non-trivial geometric features of a plane wave in a generally-covariant way. We will come back to this issue in the context of the corresponding quantum systems in section 2.3.

2.2 ROSEN AND BRINKMANN QUANTUM STATES

For these simple classical systems, the quantum dynamics is evidently determined completely by the classical dynamics, i.e. can be expressed in terms of solutions to the classical equations of motion. At first this seems to favour the Heisenberg picture, but it turns out that, in order to disentangle the time-dependent canonical transformation from the time-evolution, the Schrödinger picture is marginally more convenient and provides some more insights into the properties of the Rosen quantum system. Moreover, the proposal of [25], which is the subject of section 3, is phrased in terms of the (WKB-exact) WKB wave function of the Brinkmann system. For both these reasons, we will work in the Schrödinger picture.

As generically both the Brinkmann and the Rosen Hamiltonians are time-dependent, we will then primarily be interested in the solutions to the time-dependent Schrödinger equations

$$i\partial_t\psi_t(z) = \hat{H}_{bc}\psi_t(z) \quad i\partial_t\psi'_t(x) = \hat{H}_{rc}\psi'_t(x) \quad (2.7)$$

and in the time-dependence of expectation values of suitably chosen operators. Since the canonical transformation between the Brinkmann and Rosen systems is linear (and we are dealing with a finite-dimensional quantum system), on general grounds there exists a unitary transformation

$$U : \mathcal{H}_{rc} \rightarrow \mathcal{H}_{bc} : \quad U\psi' = \psi \quad (2.8)$$

from the Rosen to the Brinkmann space of states implementing this canonical transformation at the quantum level and mapping solutions of (2.7) into each other.⁵ Its explicit form will not be needed in the following but is given in Appendix A.2 for completeness' sake.

Turning now first to the Rosen system, the Schrödinger equation is

$$i\partial_t \psi'_t(x) = \hat{H}_{rc} \psi'_t(x) = \frac{1}{2e(t)^2} \hat{p}_x^2 \psi'_t(x) \quad (2.9)$$

In terms of the Rosen conformal-time coordinate $T(t)$ (see Appendix A.1), (2.9) reduces to the Schrödinger equation of free particle in flat space,

$$dT = dt/e(t)^2 \quad \Rightarrow \quad i\partial_T \psi'_T(x) = \frac{1}{2} \hat{p}_x^2 \psi'_T(x) \quad (2.10)$$

Thus a convenient and simple basis of solutions to (2.9) is provided by the Rosen momentum states

$$\psi'_{t,k}(x) = e^{-(ik^2/2)T(t) + ikx} \quad (2.11)$$

(these are evidently eigenstates of \hat{p}_x with eigenvalue k , delta-function normalised in the usual way). Note that the fact that the transformation $t \rightarrow T(t)$ maps the Rosen model evolution to that of a free particle in flat space already shows that in general such a description cannot be valid globally (and this reflects the Rosen coordinate singularities).

The time-dependent Schrödinger equation of the Brinkmann system

$$i\partial_t \psi_t(z) = \hat{H}_{bc} \psi_t(x) = \frac{1}{2}(\hat{p}_z^2 + \omega(t)^2 z^2) \psi_t(z) \quad , \quad (2.12)$$

on the other hand, is simply that of a time-dependent harmonic oscillator. Since we have a free (quadratic) theory, the path integral can be explicitly calculated,

$$\langle z_f | \hat{T} e^{-(i/\hbar) \int_{t_i}^{t_f} dt \hat{H}(t)} | z_i \rangle = \frac{1}{\sqrt{2\pi i \hbar (t_f - t_i)}} \sqrt{\frac{\text{Det}[-\partial_t^2]}{\text{Det}[-\partial_t^2 - \omega(t)^2]}} e^{(i/\hbar) S[z_c]} \quad . \quad (2.13)$$

Here $z_c(t)$ now denotes the classical harmonic oscillator solution with the given boundary condition, $z_c(t_i) = z_i$ and $z_c(t_f) = z_f$, $S[z_c]$ is the classical action, and the fluctuation determinants are to be calculated for zero (Dirichlet) boundary conditions.

Both the 1-loop fluctuation determinant and the classical action can be compactly expressed in terms of the Gelfand-Yaglom (GY) function (see e.g. [42] and references therein), the solution of the classical equations of motion characterised by

$$(\partial_t^2 + \omega(t)^2) F_{t_i}(t) = 0 \quad F_{t_i}(t_i) = 0 \quad \dot{F}_{t_i}(t_i) = 1 \quad (2.14)$$

⁵The notation throughout will be such that wave functions in Brinkmann coordinates (Brinkmann states) will be denoted by $\psi(z)$, perhaps with other labels indicating individual states or a basis of states, while Rosen states will be denoted by $\psi'(x)$. We will not keep track of factors of 2π in the normalisation of the wave functions.

(and its dual $G_{t_i}(t) = -\partial_{t_i} F_{t_i}(t)$, a linearly independent solution characterised by $G_{t_i}(t_i) = 1$, $\dot{G}_{t_i}(t_i) = 0$, with Wronskian $W(F, -G) = +1$). In terms of any linearly independent pair $e_{1,2}(t)$ of the classical equations of motion, the GY solution can be constructed as

$$F_{t_i}(t) = \frac{e_1(t_i)e_2(t) - e_2(t_i)e_1(t)}{W(e_1, e_2)} . \quad (2.15)$$

For the classical action one then has the expression

$$S[z_c] = \frac{1}{2F_{t_i}(t)} [\dot{F}_{t_i}(t)z^2 + G_{t_i}(t)z_i^2 - 2zz_i] , \quad (2.16)$$

and for the ratio of fluctuation determinants one has the remarkably simple result

$$\frac{1}{\sqrt{2\pi i \hbar (t_f - t_i)}} \sqrt{\frac{\text{Det}[-\partial_t^2]}{\text{Det}[-\partial_t^2 - \omega(t)^2]}} = \frac{1}{\sqrt{2\pi i \hbar F_{t_i}(t_f)}} . \quad (2.17)$$

The above-mentioned properties of the GY function and its dual imply that it also governs the time-evolution of the classical system, since one has

$$\begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} G_{t_i}(t) & F_{t_i}(t) \\ \dot{G}_{t_i}(t) & \dot{F}_{t_i}(t) \end{pmatrix} \begin{pmatrix} z(t_i) \\ \dot{z}(t_i) \end{pmatrix} \quad (2.18)$$

The amplitude (2.13) therefore gives rise to the family of Brinkmann propagator (or WKB) states

$$\psi_{t, z_i}(z) \sim \frac{1}{\sqrt{F_{t_i}(t)}} e^{iS[z_c(z, z_i; t, t_i)]} \quad (2.19)$$

characterised by $\lim_{t \rightarrow t_i} \psi_{t, z_i}(z) = \delta(z - z_i)$. Thus the GY function compactly encodes much of the relevant information about the quantum system, and for this reason we will focus on the behaviour of the GY function (and expectation values of certain operators) in our analysis of singular systems in section 3.

2.3 BRINKMANN VS ROSEN EXPECTATION VALUES

Turning now to expectation values of quantum operators, a particularly simple (and easy to work with) basis of states is provided by the Rosen momentum states $\psi'_{t,k}$ (2.11). Thus we consider general wave packets

$$\psi'_t(x) = \int dk a(k) \psi'_{t,k}(x) \quad (2.20)$$

(with the assumption that the $a(k)$ are square integrable and differentiable) and their Brinkmann duals $\psi = U\psi'$. For position observables, Rosen and Brinkmann expectation values are connected in a simple way, $U^{-1}\hat{z}U = e(t)\hat{x}$ (A.12) implying relations like

$$\langle \psi | \hat{z}^2 | \psi \rangle = e(t)^2 \langle \psi' | \hat{x}^2 | \psi' \rangle . \quad (2.21)$$

Moreover, since the Rosen momentum states are eigenstates of the momentum operator \hat{p}_x , it is straightforward to determine the expectation value of the Rosen Hamiltonian in such a wave packet,

$$\langle \psi' | \hat{H}_{rc} | \psi' \rangle = \frac{1}{2e(t)^2} \int dk k^2 a(k)^* a(k) . \quad (2.22)$$

We note that if the wave packet $\psi'_{t,x}$ is to be a solution of the Schrödinger equation, the coefficients $a(k)$ need to be time-independent. Thus we conclude that the time-dependence of the expectation value of the Rosen Hamiltonian is determined entirely by the prefactor $e(t)^{-2}$ of the Hamiltonian. In particular, if $e(t)$ has a zero somewhere, $e(t_0) = 0$, the expectation value of the Rosen Hamiltonian diverges as $t \rightarrow t_0$ for any solution to the Schrödinger equation. We will come back to this and related issues below.

In order to calculate (and compare this with) the expectation value of \hat{H}_{bc} , one needs to take into account the non-trivial relationship between the classical Hamiltonians provided by the generating function (A.8). Relegating the details of the calculations to the appendix, the results for the expectation values of \hat{H}_{rc} , \hat{x}^2 and their Brinkmann counterparts can be summarised as

$$\begin{aligned}\langle \psi' | \hat{x}^2 | \psi' \rangle &= \mathbb{A} + T(t)\mathbb{C} + T(t)^2\mathbb{B} \\ \langle \psi | \hat{z}^2 | \psi \rangle &= e(t)^2\mathbb{A} + \tilde{e}(t)^2\mathbb{B} + e(t)\tilde{e}(t)\mathbb{C} \\ \langle \psi' | \hat{H}_{rc} | \psi' \rangle &= \frac{1}{2}e(t)^{-2}\mathbb{B} \\ \langle \psi | \hat{H}_{bc} | \psi \rangle &= E(e)(t)\mathbb{A} + E(\tilde{e})(t)\mathbb{B} + E(e, \tilde{e})(t)\mathbb{C}\end{aligned}\tag{2.23}$$

where the (constant) coefficients $\mathbb{A} \neq 0, \mathbb{B} \neq 0, \mathbb{C}$, defined in (A.22), depend only on the modes $a(k)$ of the wave packet, and the functions

$$E(e) = \frac{1}{2}(\dot{e}^2 + \omega^2 e^2) \quad \text{and} \quad E(\tilde{e}) = \frac{1}{2}(\dot{\tilde{e}}^2 + \omega^2 \tilde{e}^2)\tag{2.24}$$

can be readily interpreted as the classical energies of the solutions $e(t)$ and $\tilde{e}(t)$. We have also introduced $E(e, \tilde{e}) = 1/2(\dot{e}\dot{\tilde{e}} + \omega^2 e\tilde{e})$ for uniformity of notation, and we have used some identities satisfied by $e(t)$ and its dual linearly independent solution $\tilde{e}(t) = e(t)T(t)$ to put the results for the Brinkmann expectation values into a form which makes it manifest that it is symmetric under the exchange $e(t) \leftrightarrow \tilde{e}(t)$.⁶

None of the results (2.23) should come as the slightest surprise since they just mirror the time-dependence of the corresponding general classical solution, a linear combination of $e(t)$ and its dual $\tilde{e}(t) = e(t)T(t)$. In particular, this gives the expected result for the expectation value of the Hamiltonian of the usual time-independent harmonic oscillator with constant frequency ω . In this case one has e.g. $e(t) = \sin \omega t$ and $\tilde{e}(t) = -\omega^{-1} \cos \omega t$ so that

$$\begin{aligned}e(t) = \sin \omega t &\Rightarrow E(e) = \frac{1}{2}\omega^2, \quad E(\tilde{e}) = \frac{1}{2}, \quad E(e, \tilde{e}) = 0 \\ &\Rightarrow \langle \psi | \hat{H}_{bc} | \psi \rangle = \frac{1}{2} \int d\ell [\omega^2 |a'(\ell)|^2 + \ell^2 |a(\ell)|^2]\end{aligned}\tag{2.25}$$

For the ground state wave function, with $a(\ell) \sim \exp(-\ell^2/2\omega)$, one finds $\langle \psi | \hat{H}_{bc} | \psi \rangle = \omega/2$.

Note that the corresponding Rosen Hamiltonian, on the other hand, fails completely to exhibit this perfectly regular behaviour of the time-independent harmonic oscillator. As noted before, the expectation value of the Rosen Hamiltonian diverges for all states as $t \rightarrow t_0$ with $e(t_0) = 0$, and for any solution of the harmonic oscillator equations there is an infinite number of such t_0 .

⁶Recall that, unlike the Rosen system, the Brinkmann system depends only on the frequency $\omega(t)$, not on a particular classical solution $e(t)$, and this should be (and is of course) reflected in the expectation values etc.

Geometrically, this is a pure coordinate singularity of the Rosen metric, and the quantum system fails to detect that this is not an honest singularity. The problem is not so much the Rosen states but the operators whose expectation values one wants to evaluate. To highlight this, consider the statements

$$\langle \psi | \hat{H}_{bc} | \psi \rangle = \langle \psi' | (\hat{H}_{bc})_{rc} | \psi' \rangle \quad \langle \psi' | \hat{H}_{rc} | \psi' \rangle = \langle \psi | (\hat{H}_{rc})_{bc} | \psi \rangle \quad (2.26)$$

that follow from (A.16).⁷ In the time-independent case, say, of a harmonic oscillator with constant real frequencies, the Brinkmann expectation value of the Brinkmann Hamiltonian is of course well-defined and non-singular in every Schrödinger state. Therefore, by the above identity, also the expectation value of the corresponding Rosen operator $(\hat{H}_{bc})_{rc}$ is non-singular and well-defined in every one of the corresponding Rosen states - it is a good Rosen operator. The Rosen Hamiltonian \hat{H}_{rc} , on the other hand, is not: it has a divergent expectation value in any Rosen state, and its Brinkmann-picture counterpart $(\hat{H}_{rc})_{bc}$ correspondingly has a divergent expectation value in every Brinkmann state.

The fact that the Brinkmann versus Rosen issue is a question of operators and not of states is also reinforced by the space-time origin and interpretation of these models, which provide additional physical criteria allowing one to select the operators one attributes physical significance to, thus going beyond the mere quantum mechanical formalism.

As an example, let us consider the Rosen expectation value $\langle \hat{x}^2 \rangle$. This gives the mean value of the Rosen coordinate-distance squared in some state. Without any further physical input, one can argue until one is blue in the face whether this is the physically relevant quantity to look at or whether one should rather look at $\langle \hat{z}^2 \rangle$. Indeed, in principle nothing stops one from constructing a non-relativistic quantum system with a Rosen Hamiltonian, and with the Rosen coordinates x^i Cartesian flat space coordinates, and in this case indeed it would be $\langle \hat{x}^2 \rangle$ that is the relevant quantity that can be related to observations and measurements of the system.

However, recalling the origin of the Rosen and Brinkmann Hamiltonians as the lightcone Hamiltonians of relativistic particles in the plane wave background, albeit in different coordinate systems, the situation presents itself in a rather different light. Indeed, in this case physical systems detect and measure not coordinate distance but generally-covariant quantities like proper distance, and this can easily be accounted for,

$$\langle (\text{proper Rosen distance})^2 \rangle = \langle e(t)^2 \hat{x}^2 \rangle \quad (2.27)$$

But according to (2.21), this is just the Brinkmann expectation value of \hat{z}^2 ,

$$\langle \psi' | (\text{proper Rosen distance})^2 | \psi' \rangle = \langle \psi | \hat{z}^2 | \psi \rangle \quad (2.28)$$

In this case, even if one starts in the Rosen system one is invariably led to the Brinkmann expectation value as the physically well-defined quantity to look at.

Since the time-evolution of this physically meaningful notion of distance is governed by the Brinkmann Hamiltonian and not by the Rosen Hamiltonian ($d\hat{z}/dt \sim [\hat{H}_{bc}, \hat{z}]$), this shows that

⁷ $(\hat{H}_{bc})_{rc}$ denotes the Brinkmann Hamiltonian expressed in terms of the Rosen operators \hat{x}, \hat{p}_x etc. Note that this is not the same as the Rosen Hamiltonian \hat{H}_{rc} , cf. (A.17) for the relation between the two.

it is also the Brinkmann and not the Rosen Hamiltonian that is a physically meaningful operator in the quantum theory of a relativistic particle.

We will see below that this conclusion can be significantly strengthened in that the Brinkmann dynamics is also singled out in the dynamics of strings and fields by the requirement of unitarity (and that the Rosen dynamics cannot be unitarily implemented on any field theory Fock space).

2.4 UNITARITY AND UNIQUENESS AT THE QFT/STRING LEVEL (BRINKMANN WINS HANDS DOWN)

So far we have studied aspects of the quantum mechanics of point particles in plane wave backgrounds. When one considers perturbative string theory, the relevant dynamics of closed strings (in the lightcone gauge) is that of the transverse string modes on the cylinder $S^1 \times \mathbb{R}$, and it is well known that in Brinkmann variables this reduces to the standard free action of massive scalar fields with time-dependent mass matrix $-A_{ab}(t)$ (the frequency/profile of the Brinkmann plane wave). The same is true for non-perturbative matrix string theory: in [24] it was shown that the near-singularity $t \rightarrow 0$ evolution of the plane wave matrix big bang models of [13] is dominated by the same time-dependent mass terms, while the quartic interaction and the accompanying non-Abelian nature of the dynamics are subleading. In the isotropic case we have focussed on one is thus in both cases confronted with the problem of the quantisation of a single scalar field on $S^1 \times \mathbb{R}$ with time-dependent mass term $m^2(t) = \omega^2(t)$ (or with an infinite number of Fourier modes with masses $m_n(t)^2 = m(t)^2 + n^2$, $n \in \mathbb{Z}$).

It is well known that in the quantisation of systems with an infinite number of degrees of freedom (for which there is no analogue of the Stone - von Neumann theorem) one is confronted with various ambiguities, e.g. in the choice of field parametrisation or, for a fixed parametrisation, in the choice of (Fock) representation of the canonical commutation relations (CCRs). In the usual setting of a Poincaré-invariant field theory (of a free scalar field, say), the requirement of Poincaré invariance of the Fock vacuum selects a unique Fock representation of the CCRs on which the dynamics can be unitarily implemented. In general (in particular non-stationary) space-times there is no such unique choice, and this gives rise to phenomena like particle production etc.

When there are some spatial isometries, it is natural to require that the quantisation procedure respect these symmetries, but (a) even in spatially maximally symmetric (cosmological) situations this is not enough to uniquely fix the representation, and (b) this says nothing about the dynamics and the possibility to unitarily implement the time-evolution.

In a series of papers, motivated by cosmological considerations, Cortez et al. (see e.g. [31, 32, 33, 34, 35]) investigated whether and to which extent the joint conditions of (a) invariance under the background symmetries and (b) unitary implementation of the dynamics can be used to select and specify a unique Fock quantisation. In particular, they investigated the case of scalar fields on $S^d \times \mathbb{R}$, $d \leq 3$, with a time-dependent mass (precisely the situation we are interested in here for $d = 1$), with some rather remarkable and striking uniqueness results.

To describe these, consider a scalar field ϕ satisfying the equation

$$\ddot{\phi} - \Delta\phi + s(t)\phi = 0 \tag{2.29}$$

where Δ is the Laplacian on S^d and $s(t)$ is the mass-squared function satisfying some mild regularity condition (C^1 on the time interval I under consideration being sufficient). In particular, no condition on the sign of $s(t)$ is (or needs to be) imposed.⁸

$SO(d+1)$ -invariance is now imposed as a condition on the complex structure defining the creation and annihilation operators (\hat{a}_n, \hat{a}_n^*) , thus on the Fock representation of the CCRs (this still leaves a lot of possibilities). The requirement of unitary implementability of the dynamics is then tantamount to the condition that the Bogoliubov transformation

$$\begin{pmatrix} \hat{a}_n(t) \\ \hat{a}_n^*(t) \end{pmatrix} = \begin{pmatrix} \alpha_n(t, t_i) & \beta_n(t, t_i) \\ \beta_n^*(t, t_i) & \alpha_n^*(t, t_i) \end{pmatrix} \begin{pmatrix} \hat{a}_n(t_i) \\ \hat{a}_n^*(t_i) \end{pmatrix} \quad (2.30)$$

describing the temporal evolution $t_i \rightarrow t$ of some suitably defined instantaneous creation and annihilation operators leads to finite “particle production”, $\sum_n |\beta_n|^2 < \infty$. Then the first main result is the following:

1. For every $s(t)$ (subject to the above-mentioned regularity assumptions), up to unitary equivalence there exists a unique $SO(d+1)$ -invariant Fock representation of the CCRs (complex structure) with respect to which the dynamics is unitarily implemented. This representation is independent of $s(t)$ and, in particular, coincides with the standard Fock representation of a free massless field in Minkowski space.

Note that spatial compactness is crucial for the validity of the 2nd statement, IR problems preventing an analogous statement from being true e.g. for a free scalar field in Minkowski space, where the Fock representations (and corresponding representations of the Poincaré group) are of course not unitarily equivalent for different masses.

In the above, we started with a fixed parametrisation of the fields, the scalar field ϕ and its canonically conjugate P_ϕ (a spatial density on S^d). The second main result of Cortez et al. [35] concerns the ambiguity resulting from time-dependent linear canonical transformations (field redefinitions) of the form

$$\varphi = F(t)\phi \quad , \quad P_\varphi = P_\phi/F(t) + G(t)\sqrt{h}\phi \quad (2.31)$$

where h is the determinant of the spatial metric, $F(t)$ is different from zero everywhere and both $F(t)$ and $G(t)$ are differentiable. Such transformations are commonly performed e.g. in the context of cosmological perturbation theory where they are used to put the action for the perturbations into a suitable (canonical) form. This transformation has an obvious and simple effect on the above Bogoliubov transformation implementing the time evolution, and one can analyse the conditions arising from the square-summability of the new $|\beta_n|$. It turns out that the joint requirements of symmetry and unitarity also determine an essentially unique field parametrisation:

⁸A caveat in e.g. [33, 35] regarding such a condition for the quantisation of the zero mode sector is unnecessary; the quantum theory of a particle with potential $s(t)z^2$, say, is well-defined for any sufficiently regular $s(t)$, regardless of its sign; one should simply not try to define it via naive analytic continuation from the usual Schrödinger theory of a harmonic oscillator with $\omega^2 > 0$ as that would lead to a non-hermitian Hamiltonian.

2. The dynamics cannot be unitarily implemented in an $SO(d+1)$ -invariant Fock representation unless $F(t)$ and (for $d \neq 1$) $G(t)$ are constant.

When $d = 1$, shifts $P_\phi \rightarrow P_\phi + G(t)\phi$ of the momentum are allowed, all leading to the same unique (free massless) representation of the CCRs singled out in the first result. We will see below that, for the point transformation that arises when going from Brinkmann to Rosen coordinates, $F(t)$ constant already all by itself implies $G(t) = 0$ so that this is, in any case, not an issue.

Intuitively, the fact that there is a unique field parametrisation that allows a unitary implementation of the dynamics can be attributed to the fact that any non-trivial time-dependent reparametrisation of the field will give rise to damping/friction terms in the equations of motion which are obstructions to unitarity [35].

These results have several implications for the study of (matrix) string theory in plane wave backgrounds, in particular as regards the issue of whether to use a Brinkmann or a Rosen parametrisation of the fields, in which the scalar sector of the action schematically takes the form

$$S_{bc} = -\frac{1}{2} \int d^2\sigma (\eta^{\alpha\beta} \partial_\alpha Z^a \partial_\beta Z^b - A_{ab}(t) Z^a Z^b + \dots) \quad (2.32)$$

or

$$S_{rc} = -\frac{1}{2} \int d^2\sigma (\eta^{\alpha\beta} g_{ij}(t) \partial_\alpha X^i \partial_\beta X^j + \dots) \quad , \quad (2.33)$$

the $(+\dots)$ -term including possible interaction terms and/or couplings to background fields. Exactly as for the point particle action (Appendix A.1), up to a total derivative these two actions are related by the time-dependent scaling $X^i = e_a^i(t) Z^a$ with $g_{ij}(t) e_a^i(t) e_b^j(t) = \delta_{ab}$ (this equivalence actually extends to the full non-Abelian matrix string or 3-algebra multiple M2-brane action [43]). In the isotropic case $g_{ij}(t) = e(t)^2 \delta_{ij}$, this corresponds to the canonical transformation (A.7)

$$X = Z/e(t) \quad , \quad P_X = e(t) P_Z - \dot{e}(t) Z \quad , \quad (2.34)$$

which is precisely of the form (2.31) with $F(t) = 1/e(t)$ and $G(t) = -\dot{e}(t)$ (in particular, in this case one has $F(t) = \text{const.} \Rightarrow G(t) = 0$). Note that the Brinkmann field Z has a canonical kinetic term and therefore satisfies the canonical equations of motion (2.29), while the Rosen field X plays the role of the reparametrised field φ .

The implications of the results of Cortez et al. are now clear. We consider frequencies / mass terms $s(t) = \omega(t)^2 = -A(t)$ and a time interval I such that the assumptions about $s(t)$ are satisfied (and will comment on the implications for more singular situations in section 3).

1. In the Brinkmann parametrisation there is a unique Fock quantisation respecting the translation invariance along the worldsheet S^1 (namely the standard Fock space of a free massless scalar field) such that the dynamics is unitarily implemented.
2. There is no S^1 -invariant Fock quantisation of the Rosen system with unitary time evolution.

A priori these results do not exclude the possibility that, starting from scratch in Rosen coordinates, one could find an exotic Fock representation of the CCRs and implement unitarily the temporal evolution. We know for sure, though, that the vacuum of such a representation would not be invariant under S^1 rotations. The other alternative would be a non-Fock representation. Neither option seems palatable. We believe that this should settle once and for all the question whether one should use Brinkman or Rosen fields, and we will focus entirely on the Brinkmann systems in the following.

3 REGULARISATION OF THE DYNAMICS FOR SINGULAR PLANE WAVES

We will now consider the quantum models arising from Brinkmann plane wave metrics (A.2) with profile $A_{ab}(z^+) \sim (z^+)^{-2}$, as these were shown in [5, 6, 7] to arise generically as the Penrose limits of metrics with singularities, and to which the derivation of [17] of a matrix string model (the matrix big bang) was extended in [13]. The frequencies that typically arise from the Penrose limit procedure have the form $a(1-a)(z^+)^{-2}$ for some $a \in \mathbb{R}$ and continuing to focus on the isotropic case we will thus specifically study the models with frequency / mass terms

$$\omega^2(t) = \frac{a(1-a)}{t^2} . \quad (3.1)$$

The problem is evidently invariant under $a \rightarrow 1-a$ and we can therefore, without loss of generality, restrict to the range $a > 1/2$.⁹ We will mainly be interested in the range $a > 1$ for which $\omega^2(t)$ is negative, since it is this tachyonic behaviour that characterises the Penrose limits of string theory backgrounds with strong string coupling singularities [13] (for which a non-perturbative description such as matrix string theory thus also becomes mandatory), but many of our considerations are also valid in the weakly coupled range $1/2 < a < 1$ which was the focus of interest e.g. in [19].

We recall and reiterate here that it was argued in [24], on the basis of numerical and analytical investigations of various toy-models, that both the classical and the quantum evolution of the matrix (string) system near strong string coupling (and hence weak gauge coupling) singularities are generically driven entirely by the divergent tachyonic mass terms. In particular, the quartic interaction potential, which comes with a time-dependent coefficient $g_{YM}^2 \sim t^{2q}$ with $q > 0$, can be shown to be self-consistently small in quantum-mechanical perturbation theory in that regime and appears to (perhaps somewhat disappointingly) play no role there. Thus, in the following we can and will focus on the linear harmonic oscillator dynamics even when what we have in mind are primarily applications to matrix string theory.

3.1 NECESSITY OF REGULARISATION

The considerations of section 2.4 imply that there will be a unique Brinkmann quantisation of the system with a unitary time evolution as long as one stays away from $t = 0$ (since (3.1)

⁹The value $a = 1/2$ would require a separate treatment, because one of the classical solutions has a logarithmic $\sim t^{1/2} \log t$ rather than a power-law behaviour, but is not particularly interesting in other respects. Aspects of the $a = 1/2$ model (referred to as the Gowdy model in the cosmological context) are analysed in detail in [31].

satisfies the required regularity assumptions for $t \geq t_0 > 0$).

On the other hand, it is self-evident that the quantum theory with frequency / mass terms will not miraculously regularise the classical singularity of the theory as $t \rightarrow 0$, since the evolution of wave packets and expectation values of this WKB-exact quantum system will track that of the singular classical system. Nevertheless, we will now explicitly display some of the resulting singularities, for reference purposes, and since we will attempt to regularise these expressions below.

Starting from the fact that the two linearly independent solutions of the classical equations of motion

$$\ddot{e}(t) + a(1-a)t^{-2}e(t) = 0 \quad (3.2)$$

of the point particle system, defined for $t > 0$, can be chosen to be

$$e(t) = t^{1-a} \quad , \quad \tilde{e}(t) = t^a / (1-2a) \quad (3.3)$$

(with unit Wronskian), one finds from (2.15) that the Gelfand-Yaglom function is

$$F_{t_i}(t) = \frac{1}{1-2a} (t_i^a t^{1-a} - t_i^{1-a} t^a) \quad . \quad (3.4)$$

A priori this is not defined for t_i or t negative. Moreover, this expression evidently diverges for all $t > 0$ as $t_i \rightarrow 0$ for $a > 1$ and goes to zero for all $t > 0$ as $t_i \rightarrow 0$ for $1/2 < a < 1$, either behaviour indicating a breakdown of the quantum theory. Note that the appearance of an isolated zero of the GY function for some $t = t_f$, $F_{t_i}(t_f) = 0$, is not a fundamental problem. In view of (2.14), this just signals the existence of a zero mode of the Dirichlet problem which needs to (and can) be taken care of in any number of standard ways. What happens here is that the GY function goes to zero (or diverges) for all $t > 0$, and this is a genuinely singular behaviour.

One can also see that unitarity of the time-evolution breaks down as $t \rightarrow 0$, signalled by the divergence of the Bogoliubov coefficient $\beta_{n=0}(t, t_i)$ in (2.30) as $t_i \rightarrow 0$, leading a fortiori to a breakdown of unitarity of the full quantum field / string theory. This can be established either by transforming the GY time-evolution matrix (2.18) to an appropriate oscillator basis, or directly by a calculation of the Bogoliubov coefficient $\beta(t, t_i)$ between instantaneous eigenstates at times t and t_i .¹⁰

Finally, for Brinkmann expectation values one finds from (2.23) that schematically

$$\begin{aligned} \langle \psi | \hat{z}^2 | \psi \rangle &\sim \mathbb{A} t^{2-2a} + \mathbb{B} t^{2a} + \mathbb{C} t \\ \langle \psi | \hat{H}_{bc} | \psi \rangle &\sim \mathbb{A} t^{-2a} + \mathbb{B} t^{2a-2} + \mathbb{C} t^{-1} \quad , \end{aligned} \quad (3.5)$$

which are also all singular as $t \rightarrow 0$ (suppressed numerical factors hidden in the \sim signs making the coefficients of the potentially divergent terms zero only for $a = 0, 1$, i.e. $\omega^2(t) = 0$).

¹⁰For $a \neq 0, 1$, $\beta(t, t_i)$ turns out to have the general form “(terms $\sim t^{-a}, t^{1-a}$) \times (terms $\sim t_i^{a-1}, t_i^a$) $-(t \leftrightarrow t_i)$ ”. Thus as $t_i \rightarrow 0$, $\beta(t_1, t_2)$ diverges for any $a \neq 0, 1$ (for $a = 0, 1$ one just has a free particle).

3.2 SINGLE SCALE REGULARISATION, GELFAND-YAGLOM AND FINE-TUNING

We will now try to make (at least some of) the above expressions well defined by a regularisation of the frequency / plane wave profile (thus maintaining the (α' -exact) plane wave character of the background metric). Adopting at first the suggestion in [26] we will explore the possibility of using a single-scale regularisation of the form

$$\omega^2(t) \rightarrow \Omega_\epsilon^2(t) = \epsilon^{-2} \Omega^2(t/\epsilon) \equiv \epsilon^{-2} \Omega^2(\eta) \quad (3.6)$$

in which the scale invariance of the original background is as much as possible respected and the classical equation of motion becomes independent of ϵ ,

$$\frac{d^2}{d\eta^2} f(\eta) + \Omega^2(\eta) f(\eta) = 0 \quad . \quad (3.7)$$

In order to qualify as a regularisation, $\Omega_\epsilon^2(t)$ should be non-singular for all t , in particular also $t = 0$, as long as $\epsilon \neq 0$, and should reduce to $\omega^2(t)$ as $\epsilon \rightarrow 0$ for $t > 0$. In terms of $\Omega^2(\eta)$ this amounts to the conditions

$$\exists \lim_{\eta \rightarrow 0} \Omega^2(\eta) \equiv \Omega^2(0) \quad \text{and} \quad \Omega^2(\eta) \rightarrow a(1-a)\eta^{-2} \quad \text{for} \quad \eta \rightarrow \infty \quad (3.8)$$

We will moreover assume that for $\epsilon \rightarrow 0$ and $t \neq 0$ one reaches the same limiting frequency $a(1-a)t^{-2}$ for $t < 0$ as for $t > 0$ (the case of different left/right frequencies having already been dismissed in [26]). In that case we also have the same asymptotic behaviour of $\Omega^2(\eta)$ as $\eta \rightarrow -\infty$.

Some examples of regularisations satisfying these conditions that come to mind immediately are

$$\Omega_\epsilon^2(t) = a(1-a)(t^2 + \epsilon^2)^{-1} \quad \Leftrightarrow \quad \Omega^2(\eta) = a(1-a)(\eta^2 + 1)^{-1} \quad (3.9)$$

and its generalisations

$$\Omega^2(\eta) = a(1-a) \frac{\eta^{2n}}{(\eta^2 + 1)^{n+1}} \quad (3.10)$$

(all of which we will unfortunately have to dismiss below for not accomplishing what they are supposed to do). Note that these examples have the property that $\Omega^2(-\eta) = \Omega^2(\eta)$, so that the regularised frequency preserves the \mathbb{Z}_2 reflection symmetry $t \rightarrow -t$ of the unregularised frequency. This is a natural additional assumption, but one which will not play a significant role in the analysis below.

We can then assume (see also [28]) that we have two linearly independent and ϵ -independent solutions $f_k(\eta)$ of (3.7) with the asymptotic behaviours

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \xrightarrow{\eta \rightarrow \pm\infty} \begin{pmatrix} \alpha_\pm & \beta_\pm \\ \gamma_\pm & \delta_\pm \end{pmatrix} \begin{pmatrix} |\eta|^a \\ |\eta|^{1-a} \end{pmatrix} \quad (3.11)$$

(Wronskian conservation implying that the ratio of the determinants of the two matrices is -1). Note that the generic behaviour of a solution as $|\eta| \rightarrow \infty$ is the dominant behaviour $f \sim |\eta|^a$ (recall that we have chosen to consider the range $a > 1/2$). We will call a solution *fine-tuned* if it approaches the subleading solution $|\eta|^{1-a}$ for both $\eta \rightarrow +\infty$ and $\eta \rightarrow -\infty$. The existence of such a solution is not guaranteed. If it exists, it has the following properties:

- If a fine-tuned solution exists, it is unique (any other linearly-independent solution has to exhibit the dominant asymptotic behaviour $\sim |\eta|^a$ for a non-zero Wronskian at $\pm\infty$).
- If the regularised potential is even, then the fine-tuned solution is either even or odd (since otherwise one could construct another fine-tuned solution by taking its even or odd part).

From this pair of linearly independent solutions we can construct the Gelfand-Yaglom function

$$F_{t_i}^\epsilon(t) = W(f_1, f_2)^{-1} (f_1(t_i/\epsilon)f_2(t/\epsilon) - f_2(t_i/\epsilon)f_1(t/\epsilon)) \quad (3.12)$$

and we want to investigate its behaviour as $\epsilon \rightarrow 0$. By construction this limit is well defined if t_i, t are both positive, giving rise to (3.4). The main result is that the limit is also well defined for $t_i < 0, t > 0$ iff a fine-tuned solution exists,

$$t_i < 0, t > 0 : \quad \exists \lim_{\epsilon \rightarrow 0} F_{t_i}^\epsilon(t) \quad \Leftrightarrow \quad \exists \text{ fine-tuned solution} \quad (3.13)$$

Proof: in evaluating this limit using the asymptotic behaviour, and keeping in mind that a constant η -Wronskian $W_\eta(f_1, f_2)$ implies that the relevant t -Wronskian, $t = \epsilon\eta$, is $\sim \epsilon^{-1}$, one finds one potentially divergent contribution $\sim \epsilon^{1-2a}$ whose coefficient is $\alpha_+\gamma_- - \alpha_-\gamma_+$. But

$$\alpha_+\gamma_- - \alpha_-\gamma_+ = 0 \quad \Leftrightarrow \quad \exists A, C : \begin{cases} A\alpha_+ + C\gamma_+ = 0 \\ A\alpha_- + C\gamma_- = 0 \end{cases} \quad \Leftrightarrow \quad Af_1 + Cf_2 \text{ fine-tuned} \quad (3.14)$$

This is a reformulation of a statement in [28, 26] (where the GY function is called the compression factor). With this condition in place, one finds that the regularised GY function is finite as $\epsilon \rightarrow 0$. Assuming without loss of generality that in (3.11) it is f_2 that is the fine-tuned solution (so that $\gamma_\pm = 0$ and $\alpha_+\alpha_- \neq 0$), one finds

$$\lim_{\epsilon \rightarrow 0} F_{t_i}(t) = \frac{1}{1-2a} (q|t|^a|t_i|^{1-a} + q^{-1}|t_i|^a|t|^{1-a}) \quad , \quad (3.15)$$

where $q = \alpha_+/\alpha_-$. Whenever the fine-tuned solution f_2 can be chosen to be of definite parity (e.g. when the regularised profile is even), one has $|q| = 1$ ($q = \pm 1$ for f_2 odd/even). Note that this is quite different from a naive analytic continuation of (3.4) through $t = 0$, which (for non-integer a) would give a relative imaginary phase between the two terms, and from the result one obtains [19] from the analytic continuation of the Bessel functions appearing as solutions to the string mode equations in this background.

This raises the questions how restrictive the condition is that the regularised frequency admits a fine-tuned solution and how to characterise the regularised profiles that admit such a fine-tuned solution. We will now provide some partial answers to these questions.

We consider the range $a > 1$ for which $\omega^2(t) = a(1-a)t^{-2} < 0$. First of all, we observe that a fine-tuned solution, behaving as $|\eta|^{1-a}$ for $|\eta| \rightarrow \infty$, initially (starting from $\eta = -\infty$) grows, accelerating upwards, the unregularised solution diverging as $\eta \rightarrow 0_-$. In the regularised case, however, in order to regain the fine-tuned behaviour as $\eta \rightarrow +\infty$, the solution has to turn around all the way and bend down (decelerate). This requires that there is a region around $\eta = 0$ in which the regularised profile $\Omega_\epsilon^2(t)$ changes sign and becomes positive. A simple (and rather crude) regularisation accomplishing this can be obtained by patching the original profile outside

a given η -interval to a constant potential for $\eta \in [-\eta_0, +\eta_0]$. In terms of the step-function $H(\eta)$ this means

$$\Omega^2(\eta) = a(1-a)\eta^{-2}H(|\eta| - \eta_0) + \lambda^2 H(\eta_0 - |\eta|) \quad (3.16)$$

with λ^2 a dimensionless constant. In terms of the original variable t the second term corresponds to a potential wall of width $2\epsilon\eta_0$ and height λ^2/ϵ^2 . A standard exercise in matching conditions reveals that for any value of the parameter a this potential admits a fine-tuned solution for an appropriate choice of the parameter λ .

On the other hand, the above positivity criterion immediately rules out all the candidates (3.9) and (3.10) since for $a > 1$ they are strictly negative for all η . While there appears to be no analogous obvious (positivity or other) criterion for the range $0 < a < 1$, the range of principal interest in [26] (and also previously e.g. in [19]), it is nevertheless true that (3.9) possesses no fine-tuned solutions in that range either, as can be seen by explicitly solving the equations of motion in terms of hypergeometric functions (cf. also the comment in footnote 36 of [19]). Thus for this class of examples the claim in [28] (translated into our terminology) that generically fine-tuned solutions should exist for a discrete spectrum of values of a is confirmed in the rather un-useful way that this spectrum is empty.

This shows that it is not completely obvious how to find or determine regularisations that admit the required fine-tuned solutions. In order to side-step this issue, we turn the question on its head by starting with a fine-tuned regularisation $e^\epsilon(t)$ of the solution $e(t) = t^{1-a}$ (and thus of the Rosen metric) and determining the corresponding regularised frequency $\Omega_\epsilon^2(t)$,

$$e(t) = t^{1-a} \rightarrow e^\epsilon(t) \quad \Rightarrow \quad \Omega_\epsilon^2(t) = -\frac{\ddot{e}^\epsilon(t)}{e^\epsilon(t)} \quad (3.17)$$

For example one can consider the even and odd regularisations

$$\left. \begin{aligned} e_1^\epsilon(t) &= (t^2 + \epsilon^2)^{(1-a)/2} \\ e_2^\epsilon(t) &= t(t^2 + \epsilon^2)^{-a/2} \end{aligned} \right\} \quad \Rightarrow \quad \Omega_\epsilon^2(t) = a(1-a) \frac{t^2 - \beta_k \epsilon^2}{(t^2 + \epsilon^2)^2} \quad \left\{ \begin{aligned} \beta_1 &= 1/a \\ \beta_2 &= 3/(a-1) \end{aligned} \right. \quad (3.18)$$

Note that $\beta_k > 0$ for $a > 1$ so that these regularised frequencies display the anticipated behaviour that they change sign for sufficiently small values of η . Note also that for $t \rightarrow 0$ these solutions behave as $e_k^\epsilon(t) \sim 1, t$ respectively, leading to the two alternative Rosen forms of the (evidently non-singular) Minkowski metric in that region ($t = 0$ being a mere coordinate singularity in the latter case).

Thus these regularised frequencies $\Omega_\epsilon^2(t)$ admit fine-tuned solutions by construction. It is now easy to give more general multi-parameter families of regularised frequencies that accomplish this, simply by modifying the regularised solutions $e_k^\epsilon(t)$ by terms that are subleading as $t \rightarrow 0$ or $\epsilon \rightarrow 0$.

3.3 ADDING THE DILATON

Plane waves (more generally pp-waves) give rise to exact string theory backgrounds provided that they are e.g. supplemented by a null dilaton $\phi = \phi(z^+)$ satisfying the Einstein-dilaton equation $R_{++} = -2\partial_+ \partial_+ \phi$ where R_{++} is the only non-vanishing component of the Ricci tensor

of the string-frame metric.¹¹ In the present (scale-invariant and isotropic) case this equation reduces to

$$\ddot{\phi}(t) = 4a(a-1)t^{-2} \quad \Rightarrow \quad \phi(t) = -4a(a-1)\log|t| \quad (3.19)$$

where we have selected the solution appropriate to asymptotically weak string coupling for $a > 1$, and correspondingly a strong string coupling singularity (the general solution also includes the general solution of the homogeneous equation, i.e. an affine function of t).

Once we regularise the plane wave profile, we need to enquire if the regularised dilaton equation

$$\ddot{\phi}_\epsilon(t) = -4\Omega_\epsilon^2(t) \quad (3.20)$$

admits solutions with these fine-tuned asymptotics or if linear terms necessarily arise that give rise to infinite string coupling in the far past or future. It is only in the former case that the pair $(\Omega_\epsilon^2, \phi_\epsilon)$ can legitimately be considered to be a regularisation of the original metric-dilaton background.¹²

That this is indeed an issue can easily be seen explicitly in the toy model regularisation of (3.16) or for the regularisations given in (3.18). E.g. in the latter case the general solution of the dilaton equation is

$$\phi_\epsilon(t) = 2a(a-1) \left[(1-\beta_k)(t/\epsilon) \arctan(t/\epsilon) - \log(1 + (t/\epsilon)^2) \right] + C(t/\epsilon) + D \quad (3.21)$$

For generic values of β_k this can be arranged to have the appropriate asymptotics for either $t \rightarrow +\infty$ or $t \rightarrow -\infty$ by suitable choice of integration constant C (and with $D = -4a(a-1)\log\epsilon$), but not for both simultaneously. This is only possible if $\beta_k = 1$ (with the choice $C = 0$), and comparison with (3.18) shows that this corresponds to the values $\beta_1 = 1 \rightarrow a = 1$ (which is irrelevant) or $\beta_2 = 1 \rightarrow a = 4$. Thus when one includes the dilaton in the story, the specific regularised profile given above works only for one specific value of the parameter a (and for other values other regularisations are required).

The upshot of this is that the requirement that the regularised profile admit a fine-tuned dilaton with weak coupling asymptotics in the past and future gives a further restriction on the regularised profile beyond those arising from regularity of the GY function (and its fine-tuning issue). We will return to the issue of how relevant this requirement actually is in the concluding section 4.

3.4 NECESSITY TO GO BEYOND SINGLE SCALE REGULARISATIONS

While the above seems encouraging, so far it does not allow us to claim that there is a well-defined evolution across $t = 0$ (here we disagree, at least semantically, with [26]) and/or that expectation values of operators are now well-defined at $t = 0$. Indeed, the above analysis, in particular the requirement (3.13) that $\exists \lim_{\epsilon \rightarrow 0} F_{t_i}^\epsilon(t)$ for $t_i < 0, t > 0$ does not even address the issue what happens at $t = 0$.

¹¹In the Einstein frame, the equation is $R_{++}^\epsilon = (\partial_+ \phi)^2/2$ so all the backgrounds considered here satisfy the Einstein equations with a standard, positive-definite, lightcone energy density.

¹²The regularisation of the dilaton was also discussed in [26], but the issues that arise and are relevant in the present (strong coupling) context are in some sense the opposite of those encountered there.

Therefore let us for instance look at the $\epsilon \rightarrow 0$ limit of the GY solution $F_{t_i}^\epsilon(t)$ (3.12) when one of the arguments is 0, say $t_i = 0$, while $t > 0$. Since we have chosen the two solutions $f_k(\eta)$ of the regularised equation of motion (3.7) to be ϵ -independent, the $f_k(0)$ are finite and ϵ -independent as well. One can then immediately read off that in the limit $\epsilon \rightarrow 0$ or $t \rightarrow \infty$ the GY function is a linear combination of terms of the form $\epsilon(t/\epsilon)^a$ and $\epsilon(t/\epsilon)^{1-a}$ (the addition factor of ϵ arising, as before, from the Wronskian). As a consequence of the (single scale) structure of the regularisation, the divergence structure of the GY function for $\epsilon \rightarrow 0$ is thus exactly that of the unregularised GY function (3.4) as $t_i \rightarrow 0$,

$$\epsilon \rightarrow 0 : \quad F_{t_i=0}^\epsilon(t) \sim \epsilon^a t^{1-a}, \epsilon^{1-a} t^a \quad (3.22)$$

and the $\epsilon \rightarrow 0$ limit of the GY function $F_{t_i=0}^\epsilon(t)$ will necessarily be either 0 or ∞ for all $t > 0$. Analogously, the single scale form of the regularisation implies that for $t = 0$ the divergence structure of the would-be regularised (finite ϵ) expectation values as $\epsilon \rightarrow 0$ is identical to that of the expectation values (3.5) of the unregularised ($\epsilon = 0$) theory as $t \rightarrow 0$,

$$\begin{aligned} \epsilon \rightarrow 0 : \quad \langle \psi | \hat{z}^2 | \psi \rangle(t=0) &\sim \mathbb{A} \epsilon^{2-2a} + \mathbb{B} \epsilon^{2a} + \mathbb{C} \epsilon \\ \langle \psi | \hat{H}_{bc} | \psi \rangle(t=0) &\sim \mathbb{A} \epsilon^{-2a} + \mathbb{B} \epsilon^{2a-2} + \mathbb{C} \epsilon^{-1} . \end{aligned} \quad (3.23)$$

Likewise, the classical asymptotics (3.19) translate into the statement that the $\epsilon \rightarrow 0$ behaviour of the regularised dilaton at $t = 0$ is given by $\phi_\epsilon(t=0) \sim -4a(a-1) \log |\epsilon|$.

These facts illustrate that, even at the point particle level, i.e. disregarding field-theoretic infinite particle (or rather worksheet string mode) production issues, the single scale regularisation advocated in [26] cannot be considered to give rise to a well-defined evolution from some initial time $t_i < 0$ across $t = 0$ to some final time $t_f > 0$. Nevertheless, as we will discuss in section 4, these divergences may appear in a somewhat different light when considered from the point of view of the space-time from which the plane wave arises as a Penrose limit.

The above considerations do not rule out the possibility that there are more baroque regularisations, not of the simple single-scale form (3.6), that do give finite $\epsilon \rightarrow 0$ answers even at $t = 0$, and here we show, by way of example and as a proof of concept, that this is indeed possible. We will not elaborate on this, however, since this clearly introduces even more ambiguity into the regularisation procedure.

One can for instance modify the simple regularisation (3.16) by adding a further smaller region around $\eta = 0$ with a potential well of depth $-\kappa^2$, something like

$$\Omega^2(\eta) = a(1-a)\eta^{-2}H(|\eta| - \eta_0) + \lambda^2 H(\eta_0 - |\eta|) - (\kappa^2 + \lambda^2)H(\eta_0/2 - |\eta|) . \quad (3.24)$$

Analysing the conditions for existence of a GY function that is finite and non-zero in the $\epsilon \rightarrow 0$ limit even when one of its arguments is zero, one finds that there are non-trivial solutions provided that $\kappa = \kappa(\epsilon)$ depends suitably (and non-trivially) on ϵ (which necessarily requires the introduction of new dimensionful parameters).

Once one has a well-defined non-singular GY function $F_{t_i}(t)$, one can use it and its (well-defined and non-singular) dual solution to determine expectation values from (2.23), and these are then also non-singular as $\epsilon \rightarrow 0$. However, it is not guaranteed (and may also be too much to hope

for) that also the dilaton is then regularised, in the sense that the limit $\lim_{\epsilon \rightarrow 0} \phi_\epsilon(t=0)$ exists (and this is not true e.g. for the above profile (3.24) or simple modifications thereof that we have analysed).

4 DISCUSSION

Having settled, in section 2, the issue what are the right variables to use (namely Brinkmann variables), in section 3 we analysed various aspects of the (matrix) string dynamics in the singular scale-invariant plane wave backgrounds that arise naturally in this context (e.g. from Penrose limit [5, 6] or symmetry [13] considerations).

It has become increasingly clear in recent years that a non-perturbative matrix string description in the spirit of the CSV matrix big bang model [17] is not, by itself, sufficient to resolve the null singularity of the space-time background (cf. also the assessment and discussion in [27]). For instance, as argued in [24], for the plane wave matrix big bang models of [13] near the singularity the usual quartic interaction term for the non-Abelian matrix string coordinates becomes irrelevant, due to the characteristic and inevitable presence of tachyonic mass terms in these models (for this reason, formally this argument applies to all the modes of the matrix string, not just the quantum mechanical zero mode). As a consequence, it does not appear that (in the spirit of the hope expressed in the context of the original matrix big bang model of [17]) the extra non-geometric degrees of freedom at the singularity arising from the weakly-coupled non-Abelian matrix string (which are actually also far from massless in the present case) can lead to a resolution of the singularity in this class of models. However, it would certainly be desirable to gain a better understanding of the non-linear dynamics of these models by other means.

Moreover, as far as other degrees of freedom are concerned, the decoupling arguments of [17, 13] for closed and massive open string degrees of freedom appear to be pretty robust. However, there may be some room for doubt, as the question of orders of limits becomes somewhat delicate when dealing simultaneously with Seiberg-Sen, near-singularity, and regularisation parameter $\epsilon \rightarrow 0$ limits. There may also be some subtleties with the DLCQ set-up itself for time-dependent systems (cf. the comments in [27]).

In the absence of an intrinsic regularisation mechanism or extra degrees of freedom in these models, it becomes necessary to choose some regularisation prescription in order to define the theory. While at this point any particular choice of prescription is necessarily somewhat ad hoc, one can nevertheless hope to learn something about the physics of the singularity by determining what kinds of regularisations are successful and what features they share.

A very natural first step, and one which we did not question here, is to regularise the plane wave profile itself, as this at the very least preserves the α' -exact nature of the string background required for consistency of the matrix string model. Within this setting, we investigated the proposal of [26] to consider single-scale regularisations, depending on a single dimensionful regularisation parameter ϵ , which thus additionally reflect and, as much as possible, respect, the characteristic scale-invariance of the unregularised background.

As we have shown in detail, this class of regularisations can at best be considered to give a formal asymptotic (S-matrix-like) mapping between the dynamics before and after the singularity, without, however, providing us with a regularisation of the dynamics (time-evolution of states and expectation values) through the singularity. We have also traced back the failure of this class of regularisations to the single-scale nature of the (attempted) regularisation.

We further found that allowing more complicated, multiple-scale, regularisations, i.e. modifying the profile by terms peaked around the originally singular locus and introducing additional dimensionful parameters beyond the overall scale ϵ , say, one can obtain a non-singular evolution through the singularity. We have not elaborated nor dwelled upon this, however, because it evidently opens up a lot of arbitrariness and seems to be indicating that one is missing (rather than learning) something about the physics of the singularity.

To better understand the physics of what is happening at the singularity in the models that we have analysed, recall that the plane wave metrics appearing in the plane wave matrix string models are to be considered as the lowest-order terms in a (covariant Penrose-Fermi [12]) expansion of the original (and singular) space-time metric (or string background) around its (singular) plane wave Penrose limit. Some of the divergences that one encounters appear in a different light from this perspective.

In particular, one characteristic feature of the singularities of the strong string coupling ($a > 1$) backgrounds is that particles are expelled to $z = \infty$ in finite time (and strings are infinitely stretched, without actually going through the singularity [2, 23]). A regularisation of this behaviour seems to require something novel to happen, at least some boundary condition, at $z = \infty$. However, this strongly repulsive behaviour (which, see footnote 11, is not due to the violation of some energy condition) is not necessarily something that is exhibited by the original space-time metric prior to the Penrose limit, but is actually just an artefact of the Penrose limit truncation of the metric.

Indeed, the Penrose limit procedure involves first a choice of null-geodesic and then a suitable infinite rescaling of the metric which has the effect of infinitely expanding an infinitesimal tubular neighbourhood of this null geodesic to produce the entire plane wave space-time. Thus, from this perspective, what happens when particles reach $z = \infty$ in the plane wave is that they reach the boundary of an infinitesimal neighbourhood of the geodesic in finite time, which should not be a cause of major concern.¹³

This suggests that this particular manifestation of the divergence can be cured, or at least significantly altered, by retaining higher-order terms in the Penrose-Fermi expansion. This would also allow one to move away from null singularities, and this in turn may be good news anyway, as studying null singularities in a setting in which they are rigidly null may be reason for concern because of their potential tendency to deform into spacelike singularities. However, this begs the question if or how these terms can be incorporated into the matrix string theory, and we have no answer to this at present.

¹³Note that this interpretation is not available for the weak string coupling singularities, corresponding to the parameter range $0 < a < 1$, since in this case particles and strings are infinitely squeezed and excited as they approach the singularity.

Similar statements can be made regarding the regularisation of the dilaton. We investigated the way that the dilaton depends on the regularisation, and found that the strongest constraints from dilaton regularity actually come from the elimination of diverging linear dilatons at large times t . This would be relevant if one thinks of this as a model for big bang and emergent space-time, in the spirit of [17]. However, in that case one needs to study the late-time behaviour of the non-Abelian theory [24, 44, 45]. On the other hand if, in the spirit of the Penrose limit, one considers the plane wave only as a near-singularity approximation of the full metric, then the large- t behaviour is irrelevant simply because it is also an artefact of the approximation.

More generally, and in conclusion, we feel that further work on these models should focus on (a) alternative methods to analyse and quantify the behaviour of 1+1 dimensional Yang-Mills theory with vanishing coupling, (b) perhaps a careful reanalysis of the DLCQ / decoupling arguments of [17, 13], and (c) the inclusion of higher-order terms in the Penrose-Fermi expansion in the matrix string models.

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A BRINKMANN AND ROSEN DYNAMICS

A.1 CLASSICAL BRINKMANN AND ROSEN DYNAMICS

The metric of a gravitational plane wave is

$$ds^2 = 2dz^+dz^- + A_{ab}(z^+)z^az^b(dz^+)^2 + \delta_{ab}dz^adz^b \quad (\text{A.1})$$

in Brinkmann coordinates $\{z^A\} = \{z^+, z^-, z^a\}$, and

$$ds^2 = 2dudv + g_{ij}(u)dx^idx^j \quad (\text{A.2})$$

in Rosen coordinates $\{x^I\} = \{u, v, x^i\}$. The coordinate transformation between the Rosen and Brinkmann forms of a plane wave metric has the form

$$(u, v, x^i) = (z^+, z^- + \frac{1}{2}\dot{e}_{ai}e^i_bz^az^b, e^i_az^a) \quad , \quad (\text{A.3})$$

where $e^i_a = e^i_a(u)$ is a vielbein for $g_{ij}(u)$ (satisfying the symmetry condition $M_{ab} \equiv \dot{e}_{ai}e^i_b = \dot{e}_{bi}e^i_a$), and the relation between $g_{ij}(u)$ and $A_{ab}(z^+)$ can be compactly written as [36, 20]

$$A_{ab}(z^+) = \ddot{e}_{ai}(z^+)e^i_b(z^+) \quad . \quad (\text{A.4})$$

Following [36], it is straightforward to show that $\mathbf{e} := \det(e_i^a)$ satisfies

$$\ddot{\mathbf{e}}/\mathbf{e} = \text{Tr } A + ((\text{Tr } M)^2 - \text{Tr}(M^2)) \leq \text{Tr } A = -R_{uu} \quad , \quad (\text{A.5})$$

with $R_{uu} = R_{++} = -\delta^{ab}A_{ab}$ the only non-vanishing component of the Ricci tensor of a plane wave metric. In particular, if $R_{uu} > 0$, then $\mathbf{e}(u_0) = 0$ for some finite value of u_0 , and the Rosen coordinate system breaks down there.

In the lightcone gauge $z^+ = t$ (resp. $u = t$), the geodesic Lagrangian for a particle in Brinkmann (resp. Rosen) coordinates is

$$L_{bc}(z) = \frac{1}{2}(\delta_{ab}\dot{z}^a\dot{z}^b + A_{ab}(t)z^az^b) \quad , \quad L_{rc}(x) = \frac{1}{2}g_{ij}(t)\dot{x}^i\dot{x}^j \quad . \quad (\text{A.6})$$

These two Lagrangians are equivalent in the sense that under the time-dependent coordinate transformation $x^i = e^i_a(t)z^a$ (the essential part of the transformation between Brinkmann and Rosen coordinates) they transform into each other up to a total time-derivative.

In the body of the paper we will focus on isotropic plane waves, characterised by $g_{ij}(t) = e(t)^2\delta_{ij}$. In that case, the Hamiltonian descriptions (2.4) of the systems described by the Lagrangians (2.2) are related by the linear time-dependent canonical transformation

$$(x, p_x) = (z/e(t), ep_z - \dot{e}z) \quad \Leftrightarrow \quad (z, p_z) = (e(t)x, p_x/e(t) + x\dot{e}(t)) \quad (\text{A.7})$$

that can be described by a generating function of the 2nd kind,

$$F_2(z, p_x; t) = p_x \frac{z}{e(t)} + \frac{1}{2}z^2 \frac{\dot{e}(t)}{e(t)} : \quad \left\{ \begin{array}{l} x = \frac{\partial F_2}{\partial p_x}, p_z = \frac{\partial F_2}{\partial z} \\ H_{rc} = H_{bc} + \frac{\partial F_2}{\partial t} \end{array} \right. \quad (\text{A.8})$$

Given a specific Rosen metric, specified by a solution $e(t)$ to the harmonic oscillator equation, one can construct its linearly independent Wronskian partner $\tilde{e}(t)$,

$$\tilde{e}(t) = e(t) \int_{t_0}^t \frac{dt'}{e(t')^2} \equiv e(t)T(t) : \quad W(e, \tilde{e}) \equiv e\dot{\tilde{e}} - \dot{e}\tilde{e} = 1 \quad , \quad (\text{A.9})$$

and consider the corresponding “dual” Rosen system, arising from the dual Rosen metric with $\tilde{g}_{ij}(t) = \tilde{e}(t)^2\delta_{ij}$. In the classical theory, the geometric significance of the function $T(t)$ introduced above is that of *conformal time* for the Rosen metric (isotropic plane waves are conformally flat)

$$ds^2 = 2dtdx^- + e(t)^2dx^2 = e(t)^2(2dTdx^- + dx^2) \quad . \quad (\text{A.10})$$

Its related significance in the quantum Rosen theory is explained in section 2.2.

A.2 QUANTUM BRINKMANN AND ROSEN DYNAMICS

Since classically the Brinkmann and Rosen systems are related by linear (albeit time-dependent) canonical point transformations, thus with a generating function that is at most quadratic in the canonical variables, it follows on general grounds that these canonical transformations can be unambiguously implemented by a unitary (isometric) transformation in the quantum theory.

In the case at hand, this transformation takes the form

$$\begin{aligned}\psi_t(z) &= (U\psi'_t)(z) = \frac{1}{\sqrt{e(t)}} e^{-\frac{\dot{e}(t)}{2e(t)}iz^2} \psi'_t(z/e(t)) \\ \psi'_t(x) &= (U^{-1}\psi_t)(x) = \sqrt{e(t)} e^{-\frac{1}{2}i\dot{e}(t)e(t)x^2} \psi_t(e(t)x) .\end{aligned}\tag{A.11}$$

Here the change of argument of the wave function is just (the crucial part of) the coordinate transformation between Brinkmann and Rosen coordinates, the phase factor $\exp -ik(z, t)$ is due to the fact that the two Lagrangians differ by a total time-derivative, and the prefactor $\sqrt{e(t)}$ reflects the fact that Brinkmann (resp. Rosen) states are to be normalised with respect to the measure dz (resp. $dx = dz/e(t)$).

This gives an isometry $U : \mathcal{H}_{rc} \rightarrow \mathcal{H}_{bc}$ of the Rosen and Brinkmann spaces of states (with measures dx and dz respectively) which implements the canonical transformation (A.7) in the form

$$U^{-1}\hat{z}U = e\hat{x} \quad U^{-1}\hat{p}_zU = \dot{e}\hat{x} + \frac{\hat{p}_x}{e}\tag{A.12}$$

and intertwines the action of the Schrödinger operators $i\partial_t - \hat{H}$,

$$(i\partial_t - \hat{H}_{bc})\psi_t(z) = \frac{1}{\sqrt{e(t)}} e^{-ik(z, t)} (i\partial_t - \hat{H}_{rc})\psi'_t(x) .\tag{A.13}$$

To illustrate the mapping (A.11) between Rosen and Brinkmann states, we remark that the Brinkmann propagator (WKB) state (2.19) remains a solution of the Schrödinger equation if one replaces $(F, -G)$ by any pair of solutions $(e(t), \tilde{e}(t) = e(t)T(t))$ with Wronskian $+1$, and that this more general solution

$$\psi_{t, z_i}(z) = \frac{1}{\sqrt{e(t)}} e^{(i/2)[(\dot{e}(t)/e(t))z^2 - T(t)z_i^2 - 2zz_i/e(t)]} ,\tag{A.14}$$

is precisely the one-parameter family of states one obtains by mapping the Rosen momentum modes $\psi'_{t, k}(x)$ (2.11) with $k = -z_i$ to Brinkmann coordinates using (A.11). Conversely, the Rosen-image of the propagator state (2.19) under U^{-1} is a momentum state of that Rosen metric whose metric is given by (the square of) the GY function (2.14), $e(t) = F_{t_i}(t)$.

In order to verify (A.13) it is useful to note that it follows from (A.12) that for any (reasonable, polynomial say) Brinkmann operator $\hat{O}_{bc} = \hat{O}(\hat{z}, \hat{p}_z)$ one has

$$U^{-1}\hat{O}_{bc}U = \hat{O}(U^{-1}\hat{z}U, U^{-1}\hat{p}_zU) = \hat{O}(e\hat{x}, \dot{e}\hat{x} + (1/e)\hat{p}_x) \equiv (\hat{O}_{bc})_{rc} ,\tag{A.15}$$

i.e. one obtains the Brinkmann operator expressed in terms of Rosen variables. In particular, at the level of expectation values, and for Brinkmann and Rosen states related by $\psi = U\psi'$ one generally has

$$\langle \psi | \hat{O}_{bc} | \psi \rangle = \langle \psi' | (\hat{O}_{bc})_{rc} | \psi' \rangle ,\tag{A.16}$$

while U also implements the classical relation $H_{bc} - H_{rc} = -\partial_t F_2$ (A.8) at the operator level,

$$\begin{aligned}U^{-1}\hat{H}_{bc}U &= (\hat{H}_{bc})_{rc} &= \hat{H}_{rc} - (\hat{F}_2)_{rc} \\ U\hat{H}_{bc}U^{-1} &= (\hat{H}_{rc})_{bc} &= \hat{H}_{bc} + (\hat{F}_2)_{bc} ,\end{aligned}\tag{A.17}$$

and at the level of expectation values,

$$\langle \psi | \hat{H}_{bc} | \psi \rangle = - \langle \psi' | \hat{H}_{rc} | \psi' \rangle = - \langle \psi' | (\hat{F}_2)_{rc} | \psi' \rangle = - \langle \psi | (\hat{F}_2)_{bc} | \psi \rangle , \quad (\text{A.18})$$

where $(\hat{F}_2)_{bc/rc}$ are the symmetrically ordered operators

$$\begin{aligned} (\hat{F}_2)_{bc} &= -\frac{1}{2}(\dot{e}/e)(\hat{p}_z \hat{z} + \hat{z} \hat{p}_z) + \frac{1}{2}(\dot{e}^2 - \omega^2 e^2) \hat{z}^2 / e^2 \\ (\hat{F}_2)_{rc} &= -\frac{1}{2}(\dot{e}/e)(\hat{p}_x \hat{x} + \hat{x} \hat{p}_x) - \frac{1}{2}(\dot{e}^2 + \omega^2 e^2) \hat{x}^2 . \end{aligned} \quad (\text{A.19})$$

In order to determine these expectation values for the Rosen wave packets (2.20), we need to calculate the expectation values of $\hat{p}_x \hat{x} + \hat{x} \hat{p}_x$ and \hat{x}^2 , the latter being of independent interest anyway. Straightforward calculations lead to

$$\langle \psi' | (\hat{p}_x \hat{x} + \hat{x} \hat{p}_x) | \psi' \rangle = \mathbb{C} + 2T(t)\mathbb{A} \quad (\text{A.20})$$

and

$$\langle \psi' | \hat{x}^2 | \psi' \rangle = \mathbb{A} + T(t)\mathbb{C} + T^2(t)\mathbb{B} , \quad (\text{A.21})$$

where

$$\mathbb{A} = \int dk |a'(k)|^2 \quad \mathbb{B} = \int dk k^2 |a(k)|^2 \quad \mathbb{C} = \int dk ik (a(k)^* a'(k) - a'(k)^* a(k)) , \quad (\text{A.22})$$

and we note for a nontrivial wave packet the terms \mathbb{A} and \mathbb{B} will be non-zero while \mathbb{C} can be zero (e.g. if all the $a(\ell)$ are real). The expectation value of \hat{z}^2 can then be deduced from (2.21) and (A.21),

$$\langle \psi | \hat{z}^2 | \psi \rangle = e(t)^2 \mathbb{A} + \tilde{e}(t)^2 \mathbb{B} + e(t) \tilde{e}(t) \mathbb{C} , \quad (\text{A.23})$$

while from (2.22) and (A.18) one finds for the expectation value of the Brinkmann Hamiltonian

$$\langle \psi | \hat{H}_{bc} | \psi \rangle = \frac{1}{2}(\dot{e}^2 + \omega^2 e^2) \mathbb{A} + \frac{1}{2}(\dot{\tilde{e}}^2 + \omega^2 \tilde{e}^2) \mathbb{B} + \frac{1}{2}(\dot{e} \dot{\tilde{e}} + \omega^2 e \tilde{e}) \mathbb{C} . \quad (\text{A.24})$$

Here we have made use of the fact that the dual solution $\tilde{e}(t)$ (A.9) satisfies

$$\tilde{e}(t) = e(t)T(t) \quad \Rightarrow \quad \dot{\tilde{e}}(t) = \dot{e}(t)T(t) + e(t)^{-1} , \quad (\text{A.25})$$

and introduced the functions $E(e) = \frac{1}{2}(\dot{e}^2 + \omega^2 e^2)$, $E(\tilde{e}) = \frac{1}{2}(\dot{\tilde{e}}^2 + \omega^2 \tilde{e}^2)$ and $E(e, \tilde{e}) = 1/2(\dot{e} \dot{\tilde{e}} + \omega^2 e \tilde{e})$.

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